ON ASYMPTOTICS OF $\Gamma_q(z)$ AS q APPROACHING 1

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Abstract. In this note we give a derivation of the asymptotic formula for the q-Gamma function as q approaching 1. This formula is valid on all the complex plan except at the poles of the Euler Gamma function.

1. Introduction

Recall that the q-Gamma function is defined as [1, 2, 3]

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(1-q)^{z-1}(q^z;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad a \in \mathbb{C},\ q \in (0,1).$$

$\Gamma_q(z)$ is fundamental to the theory of basic hypergeometric series. It is known that [1, 2, 3]

$$\lim_{q\to 1}\Gamma_q(z) = \Gamma(z),$$

where $\Gamma(z)$ is the Euler Gamma function. This fact is the link between the basic hypergeometric series and the classical hypergeometric series. All the standard textbooks on q-series presented W. Gosper's heuristic argument. A rigorous version of Gosper's argument and an alternative proof were later found by T. Koornwinder, [1, 5], but these proofs failed to indicate the speed of convergence. In [8] we presented a proof by using a q-Beta integral. In this note we will give yet another asymptotic formula valid on the entire complex plane except at poles of $\Gamma(z)$. Our proof only uses calculus for the $\Re(z) > 0$, then apply the transformation formula of $\theta_1(z;q)$ to get the case for $\Re(z) < 1$.

2. Main Results

For the sake of completeness we give a proof for the following Lemma, also see [4, 6].

Lemma 1. *Let*

$$|z| < 1,\ 0 < q < 1,$$

then

$$(z;q)_\infty = \exp\left\{-\sum_{k=1}^{\infty}\frac{1}{1-q^k}\frac{z^k}{k}\right\}. \tag{2.1}$$

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Proof. From

$$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}, \quad |z|<1$$

to get

$$\begin{aligned} \log (z,q)_\infty &= \sum_{j=0}^{\infty} \log\left(1-zq^j\right) = -\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} \frac{\left(zq^j\right)^k}{k} \\ &= -\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{j=0}^{\infty} q^{jk} = -\sum_{k=1}^{\infty} \frac{z^k}{k\left(1-q^k\right)} \end{aligned}$$

for $q \in (0,1)$, where all the logarithms are taken as their principle branches. (2.1) is obtained by taking exponentials. □

Lemma 2. *Let*

$$q = e^{-\pi\tau}, \ \tau > 0, \quad \Re(w) > 0,$$

then,

$$\left(q^{w+1};q\right)_\infty = \frac{\sqrt{2\pi} w^{w-1/2} \exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w)\left(1-e^{-\tau\pi w}\right)^{w+1/2}} \left\{1+\mathcal{O}(\tau)\right\}, \tag{2.2}$$

as $\tau \to 0^+$.

Proof. Take $z = qe^{-\tau\pi w}$ in (2.1) with $\Re(w) > 0$ to obtain

$$\left(qe^{-\tau\pi w}, q\right)_\infty = \exp\left\{-\sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k\left(1-q^k\right)}\right\}$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k\left(1-q^k\right)} &= \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k}\left\{\frac{q^k}{1-q^k} - \frac{1}{k\pi\tau} + \frac{1}{2} - \frac{k\pi\tau}{12}\right\} \\ &+ \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k}\left\{\frac{1}{k\pi\tau} - \frac{1}{2} + \frac{k\pi\tau}{12}\right\} \\ &= S + \frac{1}{\pi\tau}\sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k^2} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k} + \frac{\pi\tau}{12}\sum_{k=1}^{\infty} e^{-k\tau\pi w} \\ &= S + \frac{1}{\pi\tau}\mathrm{Li}_2\left(\exp(-\pi\tau w)\right) + \frac{1}{2}\log\left(1-e^{-\tau\pi w}\right) + \frac{\pi\tau}{12\left(\exp(\tau\pi w)-1\right)}, \end{aligned}$$

where

$$S = \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k}\left\{\frac{1}{e^{k\pi\tau}-1} - \frac{1}{k\pi\tau} + \frac{1}{2} - \frac{k\pi\tau}{12}\right\}$$

and

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^n}{n^2}, \quad |z| \le 1.$$

From [1]

$$\mathrm{Li}_2\left(z\right) = -\mathrm{Li}_2\left(1-z\right) + \frac{\pi^2}{6} - \log z \cdot \log\left(1-z\right)$$

to get

$$\begin{aligned}\mathrm{Li}_2\left(\exp(-\pi\tau w)\right) &= -\mathrm{Li}_2\left(1-\exp(-\pi\tau w)\right) + \frac{\pi^2}{6} + \pi\tau w \log\left(1-\exp(-\pi\tau w)\right) \\ &= -\pi\tau w + \frac{\pi^2}{6} + \pi\tau w \log\left(1-\exp(-\pi\tau w)\right) + \mathcal{O}\left(\tau^2\right),\end{aligned}$$

hence

$$\sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k\left(1-q^k\right)} = S - w + \frac{\pi}{6\tau} + \left(w+\frac{1}{2}\right)\log\left(1-\exp(-\pi\tau w)\right) + \frac{\pi\tau}{12\left(\exp(\tau\pi w)-1\right)} + \mathcal{O}\left(\tau\right)$$

as $\tau \to 0^+$. From [1]

$$\log\Gamma(w) = \left(w-\frac{1}{2}\right)\log w - w + \frac{\log(2\pi)}{2} + \int_0^{\infty}\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt$$

to obtain

$$\begin{aligned}&\int_0^{\infty}\left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt \\ &= \log\Gamma(w) - \left(w-\frac{1}{2}\right)\log w + w - \frac{\log(2\pi)}{2} - \frac{1}{12w}\end{aligned}$$

for $\Re(w) > 0$. Write

$$\begin{aligned}I &= \int_0^{\infty}\left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt \\ &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau}\left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt\end{aligned}$$

and

$$f(t) = \left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t},$$

then

$$f'(t) = \mathcal{O}\left(t\right)$$

for $t \to 0^+$ and

$$f'(t) = \mathcal{O}\left(\exp\left(-t\Re(w)\right)\right)$$

for $t \to +\infty$. Hence,

$$\begin{aligned}S - I &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau} dt \int_t^{k\pi\tau} f'(y)dy \\ &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau} f'(y)\int_{(k-1)\pi\tau}^{y} dtdy \\ &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau} f'(y)\left(y-(k-1)\pi\tau\right)dy,\end{aligned}$$

thus,

$$|S-I| \le \pi\tau\int_0^{\infty}\left|f'(y)\right|dy$$

and

$$S - I = \mathcal{O}(\pi\tau)$$

as $\tau \to 0^+$. Then,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k(1-q^k)} &= \log\Gamma(w) - \left(w - \frac{1}{2}\right)\log w - \frac{\log(2\pi)}{2} \\ &+ \frac{\pi\tau}{12}\left(\frac{1}{\exp(\tau\pi w) - 1} - \frac{1}{\pi\tau w}\right) + \frac{\pi}{6\tau} \\ &+ \left(w + \frac{1}{2}\right)\log(1 - \exp(-\pi\tau w)) + \mathcal{O}(\tau) \\ &= \log\Gamma(w) - \left(w - \frac{1}{2}\right)\log w - \frac{\log(2\pi)}{2} \\ &+ \frac{\pi}{6\tau} + \left(w + \frac{1}{2}\right)\log(1 - \exp(-\pi\tau w)) + \mathcal{O}(\tau)\end{aligned}$$

as $\tau \to 0^+$. □

Theorem 3. *Let $q = \exp(-\pi\tau)$ with $\tau > 0$. Then*

$$\Gamma_q(w) = \Gamma(w)\{1 + \mathcal{O}(\tau)\} \tag{2.3}$$

as $\tau \to 0^+$for $z \notin \mathbb{N} \cup \{0\}$.

Proof. For $\Re(w) > 0$, from (2.2) to get

$$(q^w; q)_\infty = \left(1 - e^{-\tau\pi w}\right)\left(qe^{-\tau\pi w}, q\right)_\infty = \frac{\sqrt{2\pi} w^{w-1/2}\exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w)\left(1 - e^{-\tau\pi w}\right)^{w-1/2}}\{1 + \mathcal{O}(\tau)\}$$

and

$$(q; q)_\infty = \frac{\sqrt{2\pi}\exp\left(-\frac{\pi}{6\tau}\right)}{\left(1 - e^{-\tau\pi}\right)^{1/2}}\{1 + \mathcal{O}(\tau)\}$$

as $\tau \to 0^+$. Hence, for $\Re(w) > 0$ we have

$$\Gamma_q(w) = \frac{(q; q)_\infty}{(1-q)^{w-1}(q^w; q)_\infty} = \Gamma(w)\left\{\frac{1 - e^{-\pi\tau w}}{w(1 - e^{-\pi\tau})}\right\}^{w-\frac{1}{2}}\{1 + \mathcal{O}(\tau)\} \tag{2.4}$$

as $\tau \to 0^+$. We get (2.3) from (2.4) and

$$\left\{\frac{1 - e^{-\pi\tau w}}{w(1 - e^{-\pi\tau})}\right\}^{w-\frac{1}{2}} = 1 + \mathcal{O}(\tau)$$

for $\Re(w) > 0$ as $\tau \to 0^+$. Recall that [7]

$$\theta_1(v|t) = 2\sum_{k=0}^{\infty}(-1)^k p^{(k+1/2)^2}\sin(2k+1)\pi v,$$

$$\theta_1(v|t) = 2p^{1/4}\sin\pi v(p^2; p^2)_\infty(p^2 e^{2\pi i v}; p^2)_\infty(p^2 e^{-2\pi i v}; p^2)_\infty$$

and

$$\theta_1\left(\frac{v}{t} \mid -\frac{1}{t}\right) = -i\sqrt{\frac{t}{i}}e^{\pi i v^2/t}\theta_1(v \mid t),$$

where $p = e^{\pi i t}$, $\Im(t) > 0$, then,

$$\left(q, q^{1+x}, q^{1-x}; q\right)_\infty = \frac{\exp\left(\frac{\pi\tau}{8} + \frac{\pi\tau x^2}{2}\right)\theta_1\left(x|\frac{2i}{\tau}\right)}{\sqrt{2\tau}\sinh\frac{\pi\tau x}{2}}$$

and

$$(q;q)_\infty^3 = \frac{\sqrt{2}\exp\left(\frac{\pi\tau}{8}\right)\theta_1'\left(0|\frac{2i}{\tau}\right)}{\pi\tau^{3/2}}$$

for $q = \exp(-\pi\tau)$ and $\tau > 0$. Hence for $x \notin \mathbb{Z}$ we have

$$\Gamma_q(1+x)\Gamma_q(1-x) = \frac{(q;q)_\infty^3}{\left(q, q^{1+x}, q^{1-x}; q\right)_\infty} = \frac{2\sinh\frac{\pi\tau x}{2}\theta_1'\left(0|\frac{2i}{\tau}\right)}{\pi\tau\exp\left(\frac{\pi\tau x^2}{2}\right)\theta_1\left(x|\frac{2i}{\tau}\right)}$$

and

$$\Gamma_q(x)\Gamma_q(1-x) = \frac{1-q}{1-q^x}\Gamma_q(1+x) = \frac{\left(e^{\pi\tau}-1\right)\theta_1'\left(0|\frac{2i}{\tau}\right)}{\pi\tau\exp\left(\frac{\pi\tau(x^2+x+2)}{2}\right)\theta_1\left(x|\frac{2i}{\tau}\right)}.$$

From

$$\frac{e^{\pi\tau}-1}{\pi\tau} = 1 + \mathcal{O}(\tau),$$

$$\theta_1'\left(0|\frac{2i}{\tau}\right) = 2\pi\exp\left(-\frac{\pi}{2\tau}\right)\{1 + \mathcal{O}(\tau)\},$$

$$\theta_1\left(x|\frac{2i}{\tau}\right) = 2\sin\pi x\exp\left(-\frac{\pi}{2\tau}\right)\{1 + \mathcal{O}(\tau)\}$$

and

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}, \quad x \notin \mathbb{Z}$$

to obtain

$$\Gamma_q(x) = \frac{\pi}{\sin\pi x}\frac{1}{\Gamma(1-x)}\{1 + \mathcal{O}(\tau)\} = \Gamma(x)\{1 + \mathcal{O}(\tau)\}$$

for $x < 1$ and $x \notin \mathbb{Z}$ as $\tau \to 0^+$. □

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